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Critical quantum chaos in 2D disordered systems with spin-orbit coupling

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Abstract. We examine the validity of the recently proposed semi-Poisson level spacing distribution function $P(S)$, which characterizes "critical quantum chaos", in 2D disordered systems with spin-orbit coupling. At the Anderson transition we show that the semi-Poisson $P(S)$ can describe closely the critical distribution obtained with averaged boundary conditions, over Dirichlet in one direction with periodic in the other and Dirichlet in both directions. We also obtain a sub-Poisson linear number variance $\Sigma_2(E) \approx \chi_0 + \chi E$, with asymptotic value $\chi \approx 0.07$. The obtained critical statistics, intermediate between Wigner and Poisson, is discussed for disordered systems and chaotic models.

PACS. 05.45.Mt Semiclassical chaos ("quantum chaos") – 71.30.+h Metal-insulator transitions and other electronic transitions – 72.15.Rn Localization effects (Anderson or weak localization)

In mesoscopic physics the effect of disorder on the electron propagation leads to the zero-temperature quantum Anderson metal-insulator transition, which arises from the competition between quantum tunelling and interference as a function of disorder. For weak disorder the electrons diffuse, due to quantum tunelling, and the system is metallic with correlated chaotic energy levels and "level-repulsion" described by Wigner statistics [1–3]. In the case of strong disorder the electrons localize in random positions, due to quantum interference, and the system becomes insulating, having non-chaotic completely uncorrelated (random) energy levels which show "level attraction" and obey ordinary Poisson statistics. In order to see the metal-insulator transition high enough space dimensionality (usually greater than 2) is required and at the critical point, which corresponds to an intermediate value of disorder, the level statistics changes from Wigner to Poisson [1,2]. The critical electrons are neither extended nor localized and it is believed that a new universal critical statistics, intermediate between Wigner and Poisson, should apply [4,5]. We aim to address the question of the critical statistics in two dimensions (2D), where a metal-insulator transition occurs in the presence of spin-orbit coupling [6].

The stationary energy levels of electrons in 2D quantum billiards (*e.g.* in the form of the stadium), with zero potential inside and infinity outside, can also display quantum chaotic behavior [7]. The analogies in the level statistical description bring together the two fields of mesoscopic physics and quantum chaos and have been exploited in the past for understanding important phenomena in both areas. In this respect, Wigner statistics was originally conjectured to apply for quantum systems with chaotic classical dynamics, since the levels resemble the eigenvalues found in appropriate random matrix ensembles, introduced long ago [7,8]. On the other hand, integrable systems correspond to Poisson statistics having completely uncorrelated (random) eigenvalues. The key question is again what happens at criticality, between chaos and integrability, similarly to the transition between metal and insulator. Recently, a new distribution was proposed to describe critical levels statistics [9,10], which contains both Wigner and Poisson features, as the main theme of what is called "critical quantum chaos". This intermediate distribution can be derived from a short range plasma model [9] and was realized in pseudo-integrable systems, such as the classically non-integrable but of zero metric entropy rational triangle billiards [9,11], and corresponds to other solvable models [12,13]. In disordered systems the intermediate distribution, named semi-Poisson, was shown to characterize critical states at the 3D metal-insulator transition [10] and the energy levels of few electrons in the presence of disorder and interactions [14].

In Figure 1 the energy levels E obtained from a 2D disordered system with spin-orbit coupling are displayed as a function of disorder W. The behavior of the levels is seen clearly to change at the critical point of the metal-insulator transition W_c, which separates chaotic levels (on the left) from non-chaotic levels (on the right). The chaotic levels are more regular (correlated) than the non-chaotic levels, which are uncorrelated (random). The "level-repulsion" effect can be seen on the chaotic levels and the "level-attraction" on the non-chaotic levels, where degeneracies exist. The displayed levels in Figure 1 are

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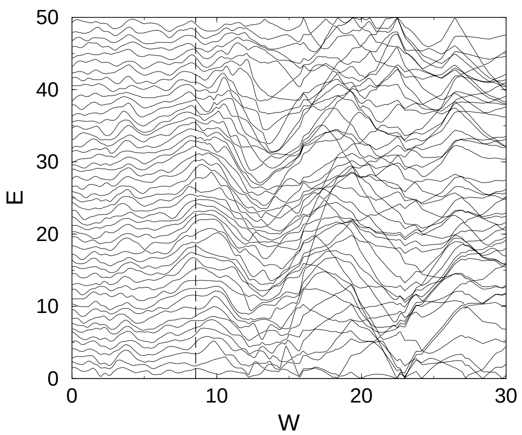

Fig. 1. Energy levels E *versus* disorder W, for a 2D disordered system with spin-orbit coupling. The metal-insulator critical point $W_c = 8.55$, marked with a broken line, separates chaotic levels (on the left) with "level-repulsion" and "spectral rigidity" described by Wigner statistics, from non-chaotic levels (on the right) with "level attraction" and "spectral randomness" described by Poisson statistics. In this paper we address what happens at the metal-insulator transition (broken line).

obtained for one random configuration from a system of linear size $L = 30$, by keeping all the energies at the middle of the spectrum, making the density of states constant at every W. The overall "spectral rigidity", seen in metallic chaotic levels, can be contrasted with the "spectral randomness" of insulating non-chaotic levels. The surprising result, also visualized in this figure, is that the chaotic levels appear more "regular" than the non-chaotic ones.

The question addressed in this paper is: "what is the level statistics at the critical broken line of Figure 1?" This study is done in connection to the scenario of "critical quantum chaos", which is summarized in a semi-Poisson level spacing distribution function $P(S)$ and a sub-Poisson linear number variance $\Sigma_2(E)$, which measures level-fluctuations in a given energy window E. Moreover, in order to examine the validity of the semi-Poisson $P(S)$ different boundary conditions (BC) must be considered, in the spirit of recent important findings [10]. The fact that the critical level fluctuations are at the same time scale-invariant (size-independent) and dependent on BC was explained by invoking the concept of the critical conductance [10]. The influence of boundary conditions on the critical level statistics has also been demonstrated for a different critical 2D model [15], without affecting the critical value of the disorder.

The main features of "critical quantum chaos" for systems characterized by the universality index $\beta = 1, 2, 4$ are summarized in: (i) The semi-Poisson $P(S)$ level spacing distribution which shows Wigner-like repulsion $\sim S^\beta$ at small spacings $S \ll 1$ and is exponential, Poisson-like, $\sim \exp(-(\beta+1)S)$ at large spacings $S \gg 1$, overall described by the scale-invariant normalized semi-Poisson curve

$$P(S) = AS^\beta \exp(-(\beta+1)S), \qquad (1)$$

with the constant values $A = 4$, $27/2$ and $3125/24$ obtained from normalization, respectively. The spacing distribution $P(S)$ is obtained by applying a "level-unfolding" procedure which keeps the level-density constant and corresponds to $\langle S \rangle = 1$. (ii) The sub-Poisson number variance, which defines the level number fluctuations in an energy window E, with the mean number proportional to E after "unfolding", according to this scenario is

$$\Sigma_2(E) \approx \chi_0 + \chi E, \qquad (2)$$

defining the level compressibility χ. The value of χ ranges between 0 (chaos) and 1 (integrability) and was related to the multifractality of the critical wavefunctions [16].

The considered disordered system displays a transition in 2D with energy levels which obey Wigner statistics for the metal (with $\beta = 4$) and Poisson statistics for the insulator (see Fig. 1). At criticality, where one expects "critical quantum chaos" to apply, numerical work suggested level-repulsion in 3D for small S [4,5], also later shown in 2D [6]. In order to study carefully the level fluctuations in the critical region it is important to identify the crucial role of BC [10]. We find that for the three considered kinds of BC the critical distribution function shows level repulsion at small spacings and is Poisson-like at large spacings. However, when considering an averaged distribution over: 1) Dirichlet BC in both directions and 2) periodic BC in one direction and Dirichlet in the other, the obtained distribution is seen to be remarkably close to the scale-invariant semi-Poisson curve of equation (1) appropriate for $\beta = 4$ (see Fig. 3).

The theoretical framework to study the Anderson transition can classify disordered systems into three universality classes, depending on whether the Hamiltonian preserves the time-reversal invariance or the rotational invariance, in direct analogy with the random matrix theory description of quantum chaotic systems [7,8]. Zero spin-orbit corresponds to the orthogonal universality class ($\beta = 1$) and finite spin-orbit to the symplectic class ($\beta = 4$), since in the last case there is time reversal symmetry but no rotational symmetry and the spin is half integer. In our calculations we consider a two-dimensional disordered system, with spin-orbit coupling for spin-$\frac{1}{2}$ particles, described by the Hamiltonian [6]

$$\mathcal{H} = \sum_{i,\sigma} \epsilon_i c^+_{i,\sigma} c_{i,\sigma} + \sum_{(i,j)} \sum_{(\sigma,\sigma')} V_{i,j;\sigma,\sigma'} c^+_{i,\sigma} c_{j,\sigma'}, \qquad (3)$$

where i labels the L^2 square lattice sites and $\sigma = \pm 1/2$ is the spin index on each site. The second sum is taken over all nearest neighbor lattice pairs (i,j) and the random on site potential ϵ_i is a spin independent uniformly distributed random variable, chosen from a probability distribution of width W. In this case the nearest neighbor hoppings $V_{i,j}$ are random 2×2 matrices describing spin rotation, due to spin-orbit, on every lattice bond (i,j). In the spinor space they are represented by

$$V_{i,j} = \begin{pmatrix} 1 + \mathrm{i}\mu V^z & \mu V^y + \mathrm{i}\mu V^x \\ -\mu V^y + \mathrm{i}\mu V^x & 1 - \mathrm{i}\mu V^z \end{pmatrix}_{ij}, \qquad (4)$$

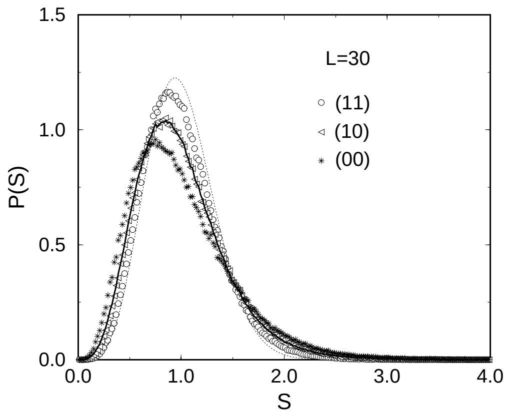

Fig. 2. This figure shows the variation of the critical $P(S)$ for three choices of BC for a system of linear size L. The mean distribution over the three cases is shown by the continuous black line. The Wigner distribution (dotted line) is also shown.

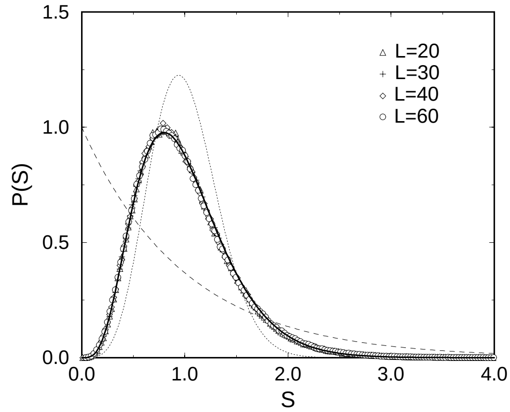

Fig. 3. The mean $P(S)$ distribution of the (10) and (00) combinations of BC for several system sizes L is shown to follow closely the semi-Poisson distribution $P(S) = (3125/24)S^4 \exp(-5S)$ (Eq. (1) for $\beta = 4$) (black line). The Wigner (dotted line) and the Poisson (dashed line) are also plotted.

where μ denotes the spin-orbit coupling and the V^x, V^y and V^z, defined for every bond (i, j), are real and independent random variables chosen from a uniform probability distribution on the interval $[-\frac{1}{2}, +\frac{1}{2}]$. For the rest the spin-orbit strength is fixed to $\mu = 2$ and the disorder is chosen to lie exactly at the critical point $W_c = 8.55$ [6].

We compute the eigenvalues from equation (3) by diagonalizing numerically the corresponding Hamiltonian matrices for large square lattices. The statistical analysis of energy levels must be done on a constant density of states using an "unfolding procedure". In order to achieve the level unfolding for the considered disordered system it is sufficient to obtain the average of the integrated density of states $\mathcal{N}$, locally at E, by repeating many times the disorder configuration creating a statistical ensemble. Then the "raw" spacings $\Delta_i = E_i - E_{i-1}$ are replaced by the 'unfolded' new ones $S_i = \mathcal{N}_{\rm av}(E_i) - \mathcal{N}_{\rm av}(E_{i-1}) \approx (E_i - E_{i-1})\frac{\partial \mathcal{N}_{\rm av}(E)}{\partial E} = (E_i - E_{i-1})/\Delta$, where Δ is the local mean spacing around E_i or equivalently the inverse density of states obtained from the raw data. In the numerical calculations we considered eigenvalues within the energy window $[-2, 2]$ performing $2700, 1200, 675$ and 794 random configuration runs in each case, with $L = 20, 30, 40$ and 60, respectively. The total number of eigenvalues from all random configurations for each BC is about 400 000 for $L = 20, 30, 40$ and 1 000 000 for $L = 60$. These "raw" data were "unfolded" in the described way.

The obtained $P(S)$ at criticality is shown in Figure 2 for the three different BC choices (11), (10) and (00), where 1 means periodic and 0 means Dirichlet BC in a given direction. The computed curves are, clearly, very different, in agreement with the corresponding 3D results for $\beta = 1$ [10]. They are very different from the Wigner or Poisson curves while their average, over the three BC, cannot fit to the semi-Poisson, either. However, the average over Dirichlet (hard wall) in both directions (00) and periodic in one direction with Dirichlet in the other (10), which is displayed in Figure 3 for various system sizes, gives a distribution very well-described by the semi-Poisson curve of equation (1) for $\beta = 4$. This is the most important result of the paper and shows the validity of the semi-Poisson for the chosen specific average over BC at criticality. The obtained semi-Poisson is also in agreement with recent results for the critical $P(S)$ at the metal-insulator transition in 3D disordered systems, where the semi-Poisson was obtained by averaging over all possible combinations of BC [10].

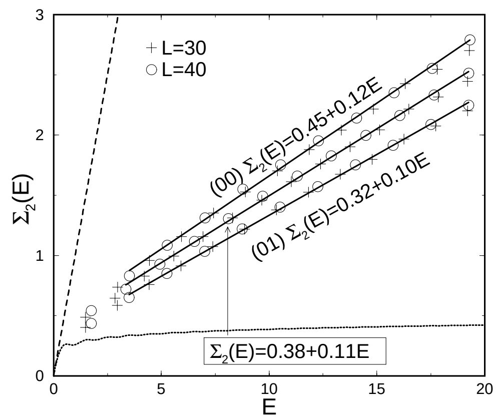

Fig. 4. The number variance $\Sigma_2(E)$ *vs.* the energy window E for the two BC (00) and (10). The straight lines fit the data giving slopes corresponding to non-asymptotic χ values. The Wigner (dotted line) and the Poisson (dashed line) are also displayed for comparison.

The longer in the E-range critical spectral fluctuations are described by a linear number variance $\Sigma_2(E) \approx \chi_0 + \chi E$, with the compressibility χ related to the critical wavefunction dimension D_2^ψ and the space dimension d *via* $\chi = (1/2)(1 - D_2^\psi/d)$ [16]. In the considered model previous studies gave $D_2^\psi \approx 1.63$ [6]. For a rather small energy window E (see Fig. 4) the level number variance is shown to be linear with level compressibility χ which varies with

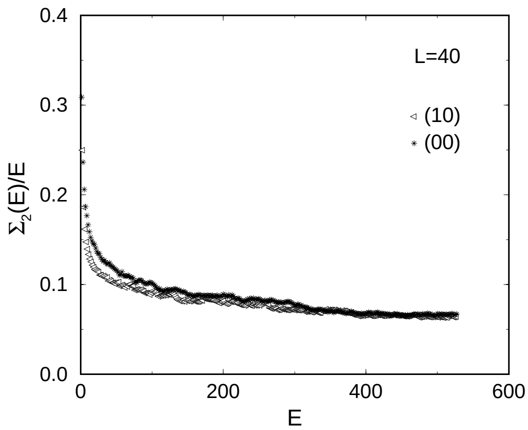

Fig. 5. $\Sigma_2(E)/E$ *versus* E for a much broader range of E where the independence on BC is seen. The asymptotic value approximates $\chi \approx 0.07$.

the chosen BC. However, when the energy window E increases the result becomes independent of BC, as was already shown in 3D [10]. The obtained asymptotic level compressibility in this case becomes $\Sigma_2(E)/E \to \chi \approx 0.07$ which leads to a $D_2^{\psi} = 1.72$, rather close to the expected value according to the previous formula (see Fig. 5). The obtained value of χ in 2D should be contrasted with the higher 3D asymptotic value $\chi \approx 0.27$ [10].

The main result from our calculations, done on a square random lattice and not on a peculiarly shaped non-random billiard, is the validity of the semi-Poisson statistics at the metal-insulator transition in 2D disordered systems with spin-orbit coupling. However, the semi-Poisson $P(S)$ is obtained for the averaged distribution over two specific BC. It must be pointed out that only the main part of the distribution agrees surprisingly well with the analytical result. For large S, where the dependence on BC should become less important, we have not succeeded to describe its tails, possibly due to their exponentially small nature. In this case the appropriate statistical measure becomes the number variance since longer range level correlations are needed. We find a linear number variance $\Sigma_2(E) \sim \chi E$ which becomes independent of the BC choice. The obtained χ is close to the expected value from the formula *via* D_2^{ψ}.

In conclusion, we have shown the validity of the semi-Poisson level statistics at the critical point of the metal-insulator transition with $\beta = 4$ in 2D. The semi-Poisson curve is shown to describe very well the main part of the computed distribution for a specific average over BC and is similar to recent results for critical disordered systems and weakly chaotic quantum systems. Our calculations, on one hand, could justify the averaging over boundary conditions recently shown to lead to the semi-Poisson statistics at the mobility edge [10]. On the other hand, suggest that such an average might be related to the bandwidth distributions, by repeating periodically the square, as it was recently shown for a non-random one-dimensional critical quasi-periodic model [17]. According to reference [18] the critical $P(S)$ with periodic boundary conditions is closer to the Poisson distribution in 4D than in 3D. Thus it might seem improbable to obtain the semi-Poisson by averaging over all BC for $\beta = 1$ in four dimensions. Clearly, more work is needed to examine the validity of the critical semi-Poisson distribution, also for 3D disordered systems with spin-orbit coupling and systems in the presence of a magnetic field ($\beta = 2$).

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